

## On Regular Linear Relations

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**Abstract** For a closed linear relation in a Banach space the concept of regularity is introduced and studied. It is shown that many of the results of Mbekhta and other authors for operators remain valid in the context of multivalued linear operators. We also extend the punctured neighbourhood theorem for operators to linear relations and as an application we obtain a characterization of semiFredholm linear relations which are regular.

**Keywords** Regular linear relation, polynomial in a linear relation, semiFredholm linear relation

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### 1 Introduction

The concept of regularity of operators in Banach spaces, was originated by Kato's classical treatment [1] of perturbation theory. Later this class of operators was studied by several authors, see for instance Mbekhta [2–4], Mbekhta and Ouahab [5] and Schmoegeer [6].

It is the purpose of this paper to consider the notion of regularity in the general setting of linear relations. It is shown that many of the results of Mbekhta and other authors for operators remain valid in the context of multivalued linear operators.

To make the paper easily accessible, some results from the theory of linear relations due to Cross [7] are recalled in Section 2. In particular, results concerning the adjoint and the minimum modulus of a linear relation and some properties of semiFredholm linear relations are presented. In Section 3 we introduce and study the class of regular linear relations. We investigate the relationship between a regular linear relation and its adjoint as well as the regularity of a polynomial in a regular linear relation with nonempty resolvent set. Section 4 contains two main theorems. The first extends the classical punctured neighbourhood theorem for operators to multivalued linear operators. This theorem leads to the notion of the jump of a semiFredholm linear relation and the second main result shows that the semiFredholm linear relations which are regular are exactly those having jump equal to zero.

Linear relations made their appearance in Functional Analysis in von Neumann [8] motivated by the need to consider adjoints of non-densely defined operators used in applications to the theory of generalized equations, [9] and also by the need to consider the inverses of certain operators, used, for example, in the study of some Cauchy problems associated with parabolic

type equations in Banach spaces [10]. Interesting works on multivalued linear operators include the treatment of degenerate boundary value problems (see, for instance, [11] and [12]), the development of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution and also to many problems of fuzzy theory (see, for instance, [13] and [14]), the application of multivalued methods to Invariant Subspace Problem [15] and [16], the application of the spectral theory of linear relations to the study of many problems of operators as, for example, the spectral theory of ordered pair of operators and of linear bundles (see, for instance, [17] and the references therein) and several papers on linear relations type semiFredholm and other classes related to them (see, for instance, [18, 19] and [7] among others). It is important to note that an investigation of semiFredholm linear relations may provide useful tools to the study of operators since the class of all bounded Fredholm operators in Banach spaces coincides with the class of inverses of closed Fredholm linear relations which are both surjective and injective.

## 2 Preliminary and Auxiliary Results

In this section we collect some auxiliary results of the theory of linear relations needed in the sequel, in the attempt of making our paper as self-contained as possible. Before beginning let us recall some basic definitions of linear relations in vector spaces following the notation and terminology of the book [7].

We will denote the set of nonnegative integers by  $\mathbb{N}$ . Let  $X$  denote a vector space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . A multivalued linear operator in  $X$  or simply a linear relation in  $X$ ,  $T : X \rightarrow X$  is a mapping from a subspace  $D(T) \subset X$ , called the domain of  $T$ , into the collection of nonempty subsets of  $X$  such that  $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$  for all nonzero  $\alpha, \beta$  scalars and  $x_1, x_2 \in D(T)$ . If  $T$  maps the points of its domain to singletons, then  $T$  is said to be a single-valued or simply an operator. We denote the class of linear relations in  $X$  by  $LR(X)$ . A linear relation in  $X$  is uniquely determined by its graph,  $G(T)$ , which is defined by  $G(T) := \{(x, y) \in X \times X : x \in D(T), y \in T x\}$ , so that we can identify  $T$  with  $G(T)$ . The inverse of  $T$  is the linear relation  $T^{-1}$  given by  $G(T^{-1}) := \{(y, x) : (x, y) \in G(T)\}$ . The subspace  $T(0)$  is called the multivalued part of  $T$ , and the subspace  $T^{-1}(0)$ , denoted by  $N(T)$ , is called the null space of  $T$  and  $T$  is called injective if  $N(T) = \{0\}$ . The range of  $T$  is the subspace  $R(T) := T(D(T))$  and  $T$  is said to be surjective if  $R(T) = X$ . Let  $M$  be a subspace of  $X$  such that  $M \cap D(T) \neq \emptyset$ . Then the linear relation  $T_M$  is given by  $G(T_M) := \{(x, y) \in G(T) : x, y \in M\}$ . For  $\lambda \in \mathbb{K}$  and  $T \in LR(X)$  the linear relations  $\lambda - T$  and  $T^2$  are defined by  $G(\lambda - T) := \{(x, \lambda x - y) : (x, y) \in G(T)\}$  and  $G(T^2) := \{(x, y) : (x, z) \in T, (z, y) \in T \text{ for some } z \in X\}$ . Hence  $T^n$ ,  $n \in \mathbb{Z}$ , is defined as usual with  $T^0 = I$  and  $T^1 = T$ . It is easily seen that  $(T^n)^{-1} = (T^{-1})^n$ ,  $n \in \mathbb{Z}$ .

Throughout the rest of this section,  $X$  will denote a normed space and  $T$  a linear relation in  $LR(X)$ , unless otherwise specified.

**Definition 1** ([7, II.3.1 and II.5.1]) *We say that  $T$  is continuous if for each neighbourhood  $V$  in  $R(T)$ , the inverse image  $T^{-1}(V)$  is a neighbourhood in  $D(T)$ , bounded if it is continuous and its domain is whole  $X$ , open if its inverse is continuous and  $T$  is called closed if its graph*

is a closed subspace of  $X \times X$ .

In order to characterize these classes of linear relations, one introduces the following notations. Let  $Q_T$  denote the quotient map from  $X$  onto  $X/\overline{T(0)}$ . It is easy to see that  $Q_T T$  is single-valued and so we can define  $\|Tx\| := \|Q_T Tx\|$ ,  $x \in D(T)$  and  $\|T\| := \|Q_T T\|$  called the norm of  $Tx$  and  $T$  respectively, and the minimum modulus of  $T$  is the quantity  $\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|, x \in D(T)\}$ .

The classes of linear relations in question may now be characterized as follows:

**Proposition 2** ([7, II.3.2 and II.5.3]) *We have*

- (i)  $T$  is continuous if and only if  $\|T\| < \infty$ .
- (ii)  $T$  is open if and only if  $\gamma(T) > 0$ .
- (iii)  $T$  is closed if and only if  $Q_T T$  is a closed operator and  $T(0)$  is a closed subspace.

We list the following useful properties of the minimum modulus of a linear relation.

**Proposition 3** ([7, II.3.4, II.3.9, II.3.11 and III.7.5]) *We have*

- (i)  $\gamma(T) \leq \gamma(Q_T T)$  with equality if  $T(0)$  is relatively closed in  $R(T)$ .
- (ii) If  $N(T)$  is closed (for example, if  $T$  is closed) and  $T$  is open, then  $N(T) = N(Q_T T)$  and  $\gamma(T) = \gamma(Q_T T)$ .
- (iii) Let  $S \in LR(X)$ . Then

$$\gamma(ST) \geq \gamma(S|_{R(T)})\gamma(T)$$

with  $\gamma(ST) = \infty$  whenever  $\gamma(T) = \infty$ . Moreover we have the implication

$$N(S) \subset R(T) \Rightarrow \gamma(ST) \geq \gamma(S)\gamma(T).$$

- (iv) Let  $T$  be open with dense range. Then for any linear relation  $S \in LR(X)$  satisfying  $S(0) \subset \overline{T(0)}$ ,  $D(T) \subset D(S)$  and  $\|S\| < \gamma(T)$ ,  $T + S$  has dense range.

Let  $M$  and  $N$  be subspaces of  $X$  and of the dual space  $X'$  respectively. We shall adopt the following notation:

$$M^\perp := \{x' \in X' : x'(M) = 0\} \quad \text{and} \quad N^\top := \{x \in X : N(x) = 0\}.$$

**Definition 4** ([7, III.1.1]) *The adjoint  $T'$  of  $T$  is defined by*

$$G(T') := G(-T^{-1})^\perp \subset X' \times X'.$$

This means that  $(y', x') \in G(T')$  if and only if  $y'(y) - x'(x) = 0$  for all  $(x, y) \in G(T)$ .

**Proposition 5** ([7, III.1.2, III.1.4, III.1.5, III.1.6 and III.4.6]) *We have*

- (i)  $T'$  is a closed linear relation in  $LR(X')$  such that

$$D(T') = \{y' \in X' : y' T \text{ is continuous and single-valued}\}.$$

- (ii) If  $S \in LR(X)$  is continuous with  $D(T) \subset D(S)$ , then  $(S + T)' = S' + T'$ .
- (iii)  $(T')^n \subset (T^n)'$ ,  $n \in \mathbb{N}$ .
- (iv)  $N(T') = R(T)^\perp$  and  $T$  is open if and only if  $N(T)^\perp = R(T')$ .

We shall make frequent use of the following result which is the multivalued version of the corresponding result for operators.

**Proposition 6** (Closed range and open mapping theorem for linear relations) *Let  $X$  be a Banach space and let  $T \in LR(X)$  be closed. The following statements are equivalent:*

- (i)  $R(T)$  is closed.
- (ii)  $R(T')$  is closed.
- (iii)  $T$  is open.
- (iv)  $T'$  is open.

*Proof* By virtue of Proposition 5, it is enough to show that  $T$  is open if and only if  $R(T)$  is closed. This last property follows upon noting that

$$T \text{ open} \Rightarrow R(T) \text{ closed (see [7, III.5.3])} \Rightarrow T^{-1} \text{ continuous (see [7, II.5.1 and III.5.4])}. \quad \square$$

Finally, certain important properties of semiFredholm linear relations are recalled.

**Definition 7** ([7, V.1.8]) *We say that  $T$  is a  $\phi_+$ -relation, denoted by  $T \in \phi_+(X)$ , if it has finite-dimensional null space and closed range, a  $\phi_-$ -relation, denoted by  $T \in \phi_-(X)$ , if its range is a closed finite-codimensional subspace of  $X$ , semiFredholm if  $T \in \phi_+(X) \cup \phi_-(X)$  and  $T$  is called Fredholm if  $T \in \phi_+(X) \cap \phi_-(X)$ . The index of a semiFredholm linear relation  $T$  is defined by  $k(T) := \alpha(T) - \beta(T)$  where  $\alpha(T) := \dim N(T)$  and  $\beta(T) := \dim X/R(T)$ .*

**Proposition 8** ([7, V.1.1, V.1.7, V.7.5, V.15.1 and V.15.6]) *Let  $X$  be a Banach space and let  $T \in LR(X)$  be closed. Then*

- (i)  $T \in \phi_+(X)$  if and only if  $T' \in \phi_-(X')$ , and  $T \in \phi_-(X)$  if and only if  $T' \in \phi_+(X')$ . Moreover if  $T$  is semiFredholm then  $\alpha(T') = \beta(T)$  and  $\alpha(T) = \beta(T')$ .
- (ii) Let  $T \in \phi_+(X)$ . Then there exists  $\epsilon > 0$  such that for every  $\lambda \in \mathbb{K}$  with  $0 < |\lambda| < \epsilon$  we have that  $\lambda - T \in \phi_+(X)$ ,  $\alpha(\lambda - T) \leq \alpha(T)$  and  $k(\lambda - T) = k(T)$ .
- (iii) Let  $T \in \phi_-(X)$ . Then there exists  $\eta > 0$  such that for every  $\lambda \in \mathbb{K}$  with  $0 < |\lambda| < \eta$ , we have that  $\lambda - T \in \phi_-(X)$ ,  $\beta(\lambda - T) \leq \beta(T)$  and  $k(\lambda - T) = k(T)$ .

Throughout the rest of the paper,  $X$  will denote a complex Banach space and  $T$  a closed linear relation in  $X$ , except where stated otherwise.

### 3 Regular Linear Relations

The following purely algebraic lemma helps to read Definition 10 below.

**Lemma 9** *Let  $T$  be a linear relation in a vector space  $X$ . The following properties are equivalent:*

- (i)  $N(T) \subset R(T^m)$  for all  $m \in \mathbb{N}$ .
- (ii)  $N(T^n) \subset R(T)$  for all  $n \in \mathbb{N}$ .
- (iii)  $N(T^n) \subset R(T^m)$  for all  $n, m \in \mathbb{N}$ .

*Proof* See [20, Lemma 3.7].  $\square$

**Definition 10** *The linear relation  $T$  is called regular if  $R(T)$  is closed, and  $T$  verifies one of the equivalent conditions of Lemma 9.*

The class of all regular linear relations in  $X$  will be denoted by  $R(X)$ .

Trivial examples of regular linear relations are surjective linear relations as well as injective linear relations with closed range.

It should be noted that if  $X$  is a Hilbert space, then the class  $R(X)$  coincides with the class of all quasi-Fredholm linear relations of degree 0 introduced in [20], in order to study the Kato decomposable linear relations in Hilbert spaces.

**Proposition 11** *If  $T \in R(X)$ , then  $\gamma(T^n) \geq \gamma(T)^n$ .*

*Proof* We proceed by induction. The cases  $n = 0, 1$  are obvious. Suppose that  $\gamma(T^n) \geq \gamma(T)^n$ . Since  $N(T) \subset R(T^n)$  (as  $T$  is regular) we conclude from Proposition 3 that  $\gamma(T^{n+1}) \geq \gamma(T)\gamma(T)^n$ . Consequently from our inductive assumption we obtain that  $\gamma(T^{n+1}) \geq \gamma(T)^{n+1}$ , which completes the proof.  $\square$

**Proposition 12** *Let  $T \in R(X)$ . Then for each  $n \in \mathbb{N}$ , we have*

- (i)  $N(T^n)^\perp = R((T')^n)$ .
- (ii)  $N((T')^n)^\top = R(T^n)$ .

*Proof* (i) The proof will be given by induction on  $n \in \mathbb{N}$ . For  $n = 0$  the statement is trivial and the case  $n = 1$  is clear from the equivalence  $T$  open if and only if  $N(T)^\perp = R(T')$  (see Proposition 5). Assume that  $N(T^n)^\perp = R((T')^n)$  and let  $x' \in N(T^{n+1})^\perp$ . Since  $N(T^n) \subset N(T^{n+1})$ ,  $x' \in N(T^n)^\perp$  and thus by the induction hypothesis there exists  $y' \in D((T')^n)$  such that  $(y', x') \in (T')^n$ . Let  $x \in R(T^n) \cap N(T)$ . Then  $(x, 0) \in T$  and  $(y, x) \in T^n$  for some  $y \in D(T^n)$ . Hence  $(y, 0) \in T^{n+1}$ , that is,  $y \in N(T^{n+1})$ ; and since  $x' \in N(T^{n+1})^\perp$  we have that  $x'(y) = 0$ . This last equality together with the facts,  $(T')^n \subset (T^n)'$  (see Proposition 5),  $(y, x) \in T^n$  and  $(y', x') \in (T')^n$  prove that  $y'(x) = 0$  and hence  $y' \in (N(T) \cap R(T^n))^\perp$ . But as  $N(T) \cap R(T^n) = N(T)$  (as  $T$  is regular),  $N(T)^\perp = R(T')$  (see again Proposition 5) and  $(y', x') \in (T')^n$  we conclude that  $x' \in R((T')^{n+1})$ . Therefore  $N(T^{n+1})^\perp \subset R((T')^{n+1})$ .

The other inclusion is clear upon noting that  $T$  is open (see Proposition 6) so that by virtue of Proposition 11,  $T^n$  is open equivalently  $N(T^n)^\perp = R((T^n)')$  (see again Proposition 5) and that the inclusion  $(T')^n \subset (T^n)'$  implies  $R((T')^n) \subset R((T^n)')$ .

(ii) The inclusion  $R(T^n) \subset N((T')^n)^\top$  is an immediate consequence of the following chain of inclusions:

$$R(T^n) \subset (R(T^n)^\perp)^\top = (N((T^n)'))^\top \subset (N((T')^n))^\top \quad (\text{see Proposition 5}).$$

The converse inclusion is proved by induction. Obviously the property is true for  $n = 0$ . Using the equality  $N(T') = R(T)^\perp$  and that  $T$  has closed range it follows that  $N(T')^\top = R(T)$  holds true. Now assume that  $(N((T')^n))^\top \subset R(T^n)$ . Let  $x \in N((T')^{n+1})^\top$  so that  $x \in N((T')^n)^\top \subset R(T^n)$  and hence there exists  $y \in D(T^n)$  such that  $(y, x) \in T^n$ . Let  $x' \in N(T^n)^\perp \cap R(T)^\perp = R((T')^n) \cap N(T')$  (by part (i) and Proposition 5). Then there exists  $y' \in D((T')^n)$  such that  $(y', x') \in (T')^n$  and  $(x', 0) \in T'$  which implies  $y' \in N((T')^{n+1})$ ; and since  $x \in (N((T')^{n+1}))^\top$  we have  $y'(x) = 0$ . But as  $(y', x') \in (T')^n \subset (T^n)'$  and  $(y, x) \in T^n$  we infer from the definition of the adjoint of  $T^n$  that  $y'(x) = x'(y)$ . In consequence  $x'(y) = 0$  and therefore  $y \in (N(T^n)^\perp \cap R(T)^\perp)^\top$ . Moreover we have the following equalities:  $(N(T^n)^\perp \cap R(T)^\perp)^\top = ((N(T^n) + R(T))^\perp)^\top = (R(T)^\perp)^\top$  (as  $T$  is regular)  $= R(T)$  (as  $T$  has closed range). Therefore  $y \in R(T)$ , and since  $x \in T^n y$  we conclude that  $x \in R(T^{n+1})$ , which completes the proof.  $\square$

We are now ready to express the first main result of this section.

**Theorem 13** *If  $T \in R(X)$  then  $T' \in R(X')$ .*

*Proof* Assume that  $T$  is regular. From Propositions 5 and 6 we infer that  $T'$  is a closed linear relation with closed range.

Let  $n, m \in \mathbb{N}$ . Then

$$\begin{aligned} N((T')^m) &\subset N((T^m)') \quad (\text{as } (T')^m \subset (T^m)') \\ &= R(T^m)^\perp \quad (\text{by Proposition 5}) \\ &\subset N(T^n)^\perp \quad (\text{by Lemma 9}) \\ &= R((T')^n) \quad (\text{by Proposition 12}). \end{aligned}$$

Hence  $T'$  is regular, as desired.  $\square$

For closed operators in Banach spaces, the above proposition was proved by Mbekhta [4, Proposition 1.6]. In [20, Lemma 6.2] it is shown that if  $X$  is a Hilbert space and  $T \in LR(X)$  is quasi-Fredholm of degree 0, then  $T$  is regular and its adjoint is also regular.

Our next aim is to show that under certain conditions, the polynomial of a regular linear relation is regular (see Theorem 21 below). To this end, we need a bit of preparation.

**Definition 14** ([21, 1]) *Let  $T$  be a linear relation in a vector space  $X$ , let  $n$  and  $m_i$ ,  $i \leq i \leq n$  be some positive integers, and let  $\lambda_i \in \mathbb{K}$ ,  $i \leq i \leq n$  be some distinct constants. Then the polynomial  $p$  in  $T$  is the linear relation*

$$p(T) := \cap_{i=1}^{i=n} (\lambda_i - T)^{m_i}.$$

The behaviour of the domain, the range, the null space and the multivalued part of  $p(T)$  is described in the following lemma which is due to Sandovici [21].

**Lemma 15** ([21, 3.2, 3.3, 3.4, 3.5 and 3.6]) *Let  $T$  be a linear relation in a vector space  $X$ , let  $n \in \mathbb{N}$ ,  $\lambda_i \in \mathbb{K}$ ,  $m_i \in \mathbb{N}$ ,  $1 \leq i \leq n$ . Assume that  $\lambda_i$ ,  $1 \leq i \leq n$  are distinct and let  $p(T)$  be as in Definition 14. Then*

- (i)  $D(p(T)) = D(T^{\sum_{i=1}^n m_i})$ .
- (ii)  $R(p(T)) = \bigcap_{i=1}^n R(\lambda_i - T)^{m_i}$ .
- (iii)  $N(p(T)) = \sum_{i=1}^n N(\lambda_i - T)^{m_i}$ .
- (iv)  $(\lambda - T)^n(0) = T^n(0)$  if  $\lambda \in \mathbb{K} \setminus \{0\}$ .
- (v)  $p(T)(0) = T^{\sum_{i=1}^n m_i}(0)$ .

**Definition 16** ([7, VI.1.2]) *Given  $T \in LR(X)$  where  $X$  is a normed space over the complex field  $\mathbb{C}$ , the resolvent set of  $T$  is the set*

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is injective, open and has dense range}\}.$$

It is clear from the closed range theorem for linear relations (Proposition 6) that if  $T$  is a closed linear relation on a complex Banach space  $X$ , then  $\rho(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is bijective}\}$ .

We recall the following property about the product of closed linear relations.

**Lemma 17** *Let  $X$  be a normed space and let  $S, T \in LR(X)$  be closed such that  $T$  has closed range and satisfies  $\alpha(T) < \infty$  and  $\gamma(T) > 0$ . Then  $ST$  is closed.*

A proof of the above lemma can be found in [7, II.5.17].

**Lemma 18** *Let  $T$  be a closed linear relation in a complex Banach space  $X$  such that  $\rho(T) \neq \emptyset$ , and let  $p(T)$  be as in Definition 14. Then  $p(T)(0)$  is closed.*

*Proof* Let  $\eta \in \rho(T)$ . Clearly  $\eta - T$  is closed and bijective and so a repeated application of Lemma 17 ensures that for each  $n \in \mathbb{N}$ ,  $(\eta - T)^n$  is closed and thus  $(\eta - T)^n(0)$  is closed (see Proposition 2). This last property and Lemma 15 imply that  $p(T)(0)$  is closed, as required.  $\square$

**Lemma 19** *Let  $T$  be open such that  $T(0)$  and  $R(T)$  are closed. Then  $T$  is closed.*

*Proof* **Case 1**  $T$  single-valued. In such case, the result is proved in [22, IV.1.6].

**Case 2**  $T$  linear relation. We first note the following simple property:

(\*) If  $M$  and  $N$  are subspaces of a normed space such that  $M$  is closed and  $M \subset N$ , then  $N$  is closed if and only if  $N/M$  is closed.

It is clear that  $R(Q_T T) = R(T)/T(0)$  (as  $T(0)$  is closed), so that  $R(Q_T T)$  is closed by virtue of (\*); and since  $\gamma(T) = \gamma(Q_T T)$  (see Proposition 3), it follows from Case 1 that  $Q_T T$  is a closed operator. The desired result now follows from Proposition 2.  $\square$

In the sequel, an important role is played by the generalized range of  $T$  which is defined by

$$R^\infty(T) := \bigcap_{n \in \mathbb{N}} R(T^n).$$

The next lemma exhibits some useful entirely algebraic properties of the generalized range of a linear relation.

**Lemma 20** *Let  $T$  be a linear relation in a vector space  $X$ . Then*

- (i) *If  $\lambda \in \mathbb{K} \setminus \{0\}$ ,  $N(\lambda - T) \subset R^\infty(T)$ .*
- (ii) *If  $\lambda, \mu \in \mathbb{K}$  are distinct, then*

$$N(\lambda - T)^n \subset R^\infty(\mu - T) \quad \text{for all } n \in \mathbb{N}.$$

- (iii) *If there exists  $d \in \mathbb{N}$  such that  $N(T) \cap R(T^d) = N(T) \cap R(T^{n+d})$  for all  $n \in \mathbb{N}$ , then*

$$T(D(T) \cap R^\infty(T)) = R^\infty(T).$$

- (iv) *If  $N(T) \subset R^\infty(T)$  or  $\dim N(T) < \infty$ , then*

$$T(D(T) \cap R^\infty(T)) = R^\infty(T).$$

*Proof* (i) See [23, Lemma 7.2].

- (ii) Since  $\lambda - T = (\lambda - \mu) + (\mu - T)$ , the assertion (ii) follows immediately from (i).

Evidently the inclusion  $T(D(T) \cap R^\infty(T)) \subset R^\infty(T)$  holds for every linear relation, so that in (iii) and (iv) we need only to verify the opposite inclusion.

(iii) Let  $y \in R^\infty(T)$ . Then for each  $n \in \mathbb{N}$ , there exists  $x_n \in D(T) \cap R(T^n)$  such that  $y \in Tx_n$ . Thus  $0 = y - y \in Tx_n - Tx_m = T(x_n - x_m)$ , that is,  $x_n - x_m \in N(T)$ ,  $n, m \in \mathbb{N}$ . This fact combined with the equalities  $N(T) \cap R(T^d) = N(T) \cap R^\infty(T) = \bigcap_{m \in \mathbb{N}} (N(T) \cap R(T^m))$

gives:  $x_{d+1} - x_d \in N(T) \cap R(T^d) \subset R(T)$ ;  $x_{d+2} - x_d \in N(T) \cap R(T^d) \subset R(T^2)$ ;  $\dots$ ;  $x_{d+n} - x_d \in N(T) \cap R(T^d) \subset R(T^n)$ ;  $\dots$ . In consequence  $x_d \in R^\infty(T)$  and since  $y \in Tx_d$  we conclude that  $y \in T(D(T) \cap R^\infty(T))$ , as desired.

(iv) Obviously if  $N(T) \subset R^\infty(T)$  then  $N(T) \cap R(T^d) = N(T) \cap R(T^{n+d}) = N(T)$  for all integers  $n, d \geq 0$ . Hence it suffices to apply (iii).

If  $N(T)$  is finite-dimensional, then the sequence  $(N(T) \cap R(T^n))$  is a stationary sequence for  $n$  large enough which proves that there exists  $d \in \mathbb{N}$  such that  $N(T) \cap R(T^d) = N(T) \cap R(T^{n+d})$  for all  $n \geq d$ . Therefore also in this case we can apply the part (iii) to obtain the desired statement.  $\square$

We are now in a position to prove the second main result of this section.

**Theorem 21** *Let  $n \in \mathbb{N}$ ,  $\lambda_i \in \mathbb{K}$ ,  $m_i \in \mathbb{N}$ ,  $1 \leq i \leq n$ . Assume that  $\rho(T) \neq \emptyset$ , that  $\lambda_i$  are distinct and that  $\lambda_i - T$  are regular,  $1 \leq i \leq n$ . Then  $p(T) = \cap_{i=1}^n (\lambda_i - T)^{m_i}$  is regular.*

*Proof* By Lemma 18, we have

$$p(T)(0) \text{ is closed.} \quad (1)$$

Since  $\lambda_i - T$  is closed for every  $i \in \{1, 2, \dots, n\}$  (as  $(\lambda_i - T)$  is regular), we obtain from Proposition 12 that  $R(\lambda_i - T)^{m_i}$  is closed, and then it follows from Lemma 15 that

$$R(p(T)) \text{ is closed.} \quad (2)$$

Now, we show that  $p(T)$  is open. To see this, we observe that by virtue of Proposition 6, each factor  $\lambda_i - T$  is open. Furthermore, we have

$$\begin{aligned} 0 &< \gamma((\lambda_i - T)^{m_i}) \quad (\text{by Proposition 11}) \\ &\leq \gamma((\lambda_i - T)^{m_i}|_{(R((\lambda_j - T)^{m_j}) + N((\lambda_i - T)^{m_i}))}) \end{aligned}$$

(since the relations  $(\lambda_i - T)^{m_i}$  and  $(\lambda_i - T)^{m_i}|_{(R((\lambda_j - T)^{m_j}) + N((\lambda_i - T)^{m_i}))}$  have the same null space)

$$= \gamma((\lambda_i - T)^{m_i}|_{R(\lambda_j - T)^{m_j}}) \quad (\text{By Lemma 20}).$$

Now, it follows from Proposition 3 that

$$\gamma((\lambda_i - T)^{m_i}(\lambda_j - T)^{m_j}) \geq \gamma((\lambda_i - T)^{m_i}|_{R(\lambda_j - T)^{m_j}})\gamma((\lambda_j - T)^{m_j}) > 0,$$

that is,  $(\lambda_i - T)^{m_i}(\lambda_j - T)^{m_j}$  is open.

Continuing in this way we obtain that

$$p(T) \text{ is open.} \quad (3)$$

Now, a combination of (1), (2), (3) and Lemma 19 gives

$$p(T) \text{ is closed.} \quad (4)$$

To prove that  $p(T)$  is regular it only remains to show that  $N(p(T)) \subset R^\infty(p(T))$ . To see this, let us suppose that  $x \in N(p(T))$ . By Lemma 15 there exists for each  $i = 1, 2, \dots, n$  an element  $x_i \in N(\lambda_i - T)^{m_i}$  such that  $x = \sum_{i=1}^n x_i$ . Since the linear relations  $\lambda_i - T$  are regular for every  $i = 1, 2, \dots, n$ ,  $x_i \in R^\infty(\lambda_i - T)$ . On the other hand  $\lambda_i \neq \lambda_j$  and so by



part (ii) of Lemma 20 we also have  $N(\lambda_i - T) \subset R^\infty(\lambda_j - T)$  so that  $x_i \in R^\infty(\lambda_j - T)$  for all  $i, j \in \{1, 2, \dots, n\}$ . Therefore,

$$x \in \bigcap_{i=1}^n R^\infty(\lambda_i - T),$$

and hence by Lemma 15,  $x \in R^\infty(p(T))$ . Therefore we conclude that  $N(p(T)) \subset R^\infty(p(T))$ , which completes the proof.  $\square$

For closed operators the above theorem was proved by Mbekhta [3, Proposition 5.4].

If  $S$  and  $T$  are linear relations in a vector space such that  $S$  and  $T$  commute, that is,  $ST = TS$  in the sense of the product of linear relations, then it is easy to see that  $(\lambda - S)T = T(\lambda - S)$  holds true for all  $\lambda \in \mathbb{K}$ , and hence applying this property to  $\eta - T$  instead of  $T$  we have  $(\lambda - S)(\eta - T) = (\eta - T)(\lambda - S)$ . Furthermore it is clear that  $S^n T^m = T^m S^n$  holds for all  $n, m \in \mathbb{N}$ , and using this equality with  $\lambda - S$  and  $\eta - T$  instead of  $S$  and  $T$  it follows that

$$(\lambda - S)^n (\eta - T)^m = (\eta - T)^m (\lambda - S)^n$$

for all  $\lambda, \eta \in \mathbb{K}$  and for all  $n, m \in \mathbb{N}$ . This property together Theorem 21 suggests to investigate the regularity of the product of commuting regular linear relations. The following example due to Müller [24], shows that the product of two commuting operators need not to be regular.

**Example 22** Let  $X$  be a Hilbert space with an orthonormal basis  $(e_{i,j})$  where  $i, j$  are integers for which  $ij \leq 0$ . Let  $T$  and  $S$  be defined by

$$\begin{aligned} T e_{i,j} &:= 0 \text{ if } i = 0, j > 0 \quad \text{and} \quad T e_{i,j} := e_{i+1,j} \quad \text{otherwise.} \\ S e_{i,j} &:= 0 \text{ if } i > 0, j = 0 \quad \text{and} \quad S e_{i,j} := e_{i,j+1} \quad \text{otherwise.} \end{aligned}$$

Then  $T$  and  $S$  are bounded regular operators such that  $TS = ST$  and  $TS$  is not regular.

We end this section with a generalization of [4, Theorem 1.9].

**Theorem 23** *Let  $T \in R(X)$ . Then there exists  $\nu > 0$  such that  $\lambda - T \in R(X)$  if  $|\lambda| < \nu$ .*

*Proof* We first note that

$$\text{If } T \in R(X) \text{ and } \lambda \in \mathbb{K}, \text{ then } \gamma(\lambda - T) \geq \gamma(T) - 3|\lambda|. \quad (5)$$

Note that  $T(D(T) \cap R^\infty(T)) = R^\infty(T)$  (see Lemma 20) and thus the proof of (5) is along the lines of the proof of the analogous result provided in [4, Lemma 1.8] for the case when  $T$  is single-valued.

Let  $T_\infty : R^\infty(T) \rightarrow R^\infty(T)$  be the restriction of  $T$  to  $R^\infty(T)$ , and let  $\lambda_\infty$  be the restriction of  $\lambda I$  to  $R^\infty(T)$ . Since  $T$  is regular, then  $R(T^n)$  is closed for every  $n \in \mathbb{N}$  by virtue of Proposition 12, so that  $R^\infty(T)$  is closed; and since  $T$  is closed,  $T_\infty$  must be closed. Using the assertion (iv) of Lemma 20,  $T_\infty$  is surjective and thus by the Open mapping theorem for linear relations (see Proposition 6),  $T_\infty$  is open, that is,  $\gamma(T_\infty) > 0$ . The statement (iv) of Proposition 3 allows us to write  $R^\infty(T) = R(\lambda_\infty - T_\infty)$  whenever  $0 < |\lambda| < \gamma(T_\infty)$ . Consequently

$$R^\infty(T) \subset R^\infty(\lambda - T) \quad \text{for all } 0 < |\lambda| < \gamma(T_\infty). \quad (6)$$

From (6) and the fact  $N(\lambda - T) \subset R^\infty(T)$  (see Lemma 20) we obtain that

$$N(\lambda - T) \subset R^\infty(\lambda - T) \quad \text{for all } 0 < |\lambda| < \gamma(T_\infty). \quad (7)$$

Hence, it only remains to verify that there exists  $0 < \nu < \gamma(T_\infty)$  such that for  $0 < |\lambda| < \nu$ ,  $\lambda - T$  has closed range, equivalently  $\lambda - T$  is open (by Proposition 6). But  $0 < \gamma(T) \leq \gamma(T|_{R^\infty(T)+N(T)}) = \gamma(T|_{R^\infty(T)})$  (as  $T$  is regular) so that

$$0 < \gamma(T) \leq \gamma(T_\infty). \quad (8)$$

Combining (5), (7) and (8) we conclude that  $\lambda - T$  is regular whenever  $|\lambda| < \gamma(T)/3$  which completes the proof.  $\square$

#### 4 Punctured Neighbourhood Theorem for Linear Relations and Regular Linear Relations

If a bounded operator  $T$  in a Banach space  $X$  is semiFredholm then the classical punctured theorem says that there exists  $\alpha_T > 0$  for which

$$\alpha(\lambda - T) \text{ is constant whenever } 0 < |\lambda| < \alpha_T \text{ if } T \in \phi_+(X),$$

and

$$\beta(\lambda - T) \text{ is constant whenever } 0 < |\lambda| < \alpha_T \text{ if } T \in \phi_-(X).$$

Our next aim is to extend this result to closed multivalued linear operators. For this, we will use Proposition 8, Lemma 20 and the following result concerning the product of  $\phi_+$ -linear relations.

**Proposition 24** *If  $T \in \phi_+(X)$ , then  $T^n \in \phi_+(X)$  for all  $n \in \mathbb{N}$ .*

*Proof* For each  $n \in \mathbb{N}$ , we have  $\alpha(T^n) < \infty$  (since  $\alpha(T^n) \leq n\alpha(T)$  [23, Lemma 5.4] and  $N(T)$  is finite-dimensional),  $T^n$  is closed (by a repeated application of Lemma 17). Therefore, it only remains to prove that  $R(T^n)$  is closed. We proceed by induction. The cases  $n = 0$  and  $n = 1$  are trivial. Assume that  $R(T^n)$  is closed, so that  $T^n \in \phi_+(X)$ . Then  $Q_{T^n}T^n$  is a closed operator (see Proposition 2) such that  $R(Q_{T^n}T^n)$  is closed,  $\dim N(Q_{T^n}T^n) < \infty$  (since  $\alpha(T^n) < \infty$  and  $N(T^n) = N(Q_{T^n}T^n)$  by Proposition 3) and hence  $R(T) + N(Q_{T^n}T^n)$  is closed. Combining these properties with [22, IV.2.9], it follows that  $Q_{T^n}T^n R(T)$  is closed. Therefore,  $R(T^{n+1})/T^n(0)$  is closed which implies that  $R(T^{n+1})$  is also closed, since  $T^n(0)$  is closed.  $\square$

**Theorem 25** (Punctured neighbourhood theorem for linear relations) *We have*

- (i) *If  $T \in \phi_+(X)$ , then there exists  $\epsilon > 0$  such that  $\lambda - T \in \phi_+(X)$  and  $\alpha(\lambda - T)$  is constant on the annulus  $0 < |\lambda| < \epsilon$ . Moreover  $\alpha(\lambda - T) \leq \alpha(T)$  for all  $|\lambda| < \epsilon$ .*
- (ii) *If  $T \in \phi_-(X)$ , then there exists  $\eta > 0$  such that  $\lambda - T \in \phi_-(X)$  and  $\beta(\lambda - T)$  is constant on the annulus  $0 < |\lambda| < \eta$ . Moreover  $\beta(\lambda - T) \leq \beta(T)$  for all  $|\lambda| < \eta$ .*

*Proof* (i) Let  $\lambda \in \mathbb{K} \setminus \{0\}$  and assume that  $T \in \phi_+(X)$ . Let  $T_\infty$  and  $\lambda_\infty$  be as in Theorem 23.

From the hypothesis,  $T$  is closed, and since  $R^\infty(T)$  is closed (since  $T^n \in \phi_+(X)$  by Proposition 24),  $T_\infty$  must be closed, and since  $T_\infty$  is surjective (see Lemma 20),  $T_\infty$  is a  $\phi_-$ -relation. Now, from Proposition 8 there exists  $\eta > 0$  for which  $\lambda_\infty - T_\infty$  is a  $\phi_-$ -relation

with  $\beta(\lambda_\infty - T_\infty) \leq \beta(T_\infty) = 0$  and  $k(\lambda_\infty - T_\infty) = k(T_\infty)$  whenever  $0 < |\lambda| < \eta$ . Consequently,  $\dim N(\lambda_\infty - T_\infty) = \dim N(T_\infty)$ , and since  $N(\lambda_\infty - T_\infty) = N(\lambda - T) \cap R^\infty(T) = N(\lambda - T)$  (see Lemma 20) and  $N(T_\infty) = N(T) \cap R^\infty(T)$  it follows that  $\alpha(\lambda - T)$  is constant on the punctured neighbourhood  $0 < |\lambda| < \eta$ .

(ii) Let  $\lambda \in \mathbb{K} \setminus \{0\}$  and let  $T \in \phi_-(X)$ . Then, the adjoint of  $T$  is a  $\phi_+$ -relation by Proposition 8, and thus by the part (i) there exists  $\epsilon > 0$  such that  $\lambda - T' \in \phi_+(X')$  and  $\alpha(\lambda - T')$  is constant on the annulus  $0 < |\lambda| < \epsilon$ . Now, the desired result follows upon noting that  $(\lambda - T)' = \lambda - T'$  (see Proposition 5) and that  $\alpha((\lambda - T)') = \beta(\lambda - T)$  (see Proposition 8).  $\square$

The above theorem leads to the notion of the jump of a semiFredholm linear relation on a Banach space, defined as follows:

**Definition 26** (i) Let  $T \in \phi_+(X)$  and let  $\epsilon > 0$  as in Theorem 25 (i). The jump of  $T$  is defined by

$$j(T) := \alpha(T) - \alpha(\lambda - T), \quad 0 < |\lambda| < \epsilon.$$

(ii) Let  $T \in \phi_-(X)$  and let  $\eta > 0$  as in Theorem 25 (ii). The jump of  $T$  is defined by

$$j(T) := \beta(T) - \beta(\lambda - T), \quad 0 < |\lambda| < \eta.$$

Clearly  $j(T) \geq 0$  and the continuity of the index (see Proposition 8) ensures that both definitions of  $j(T)$  coincide whenever  $T \in \phi_+(X) \cap \phi_-(X)$ , so that  $j(T)$  is unambiguously defined.

An immediate consequence of Proposition 8 and Theorem 25 is that if  $T$  is semiFredholm then  $j(T) = j(T')$ .

It is known (see, for example, [1, 4] and [25]) that if  $T$  is a bounded semiFredholm operator then  $T$  is regular if and only if  $j(T) = 0$ . The following result proves the validity of this last property in the context of multivalued linear operators.

**Theorem 27** Let  $T \in \phi_+(X) \cup \phi_-(X)$ . Then  $T$  is regular if and only if  $j(T) = 0$ .

*Proof* **Case 1**  $T \in \phi_+(X)$ . Arguing as in the proof of Theorem 25, there exists  $\eta > 0$  for which  $\dim N(\lambda - T) = \dim(N(T) \cap R^\infty(T))$  if  $0 < |\lambda| < \eta$ , and hence  $j(T) = 0$  if  $T$  is regular.

Assume now that  $j(T) = 0$ , namely, there exists  $\epsilon > 0$  such that  $\alpha(\lambda - T)$  is constant for  $|\lambda| < \epsilon$ . Then

$$\alpha(T_\infty) \leq \alpha(T) = \alpha(\lambda - T) = \alpha(\lambda_\infty - T_\infty) \quad \text{for all } 0 < |\lambda| < \epsilon,$$

where the last equality is true by virtue of Lemma 20.

But since  $T_\infty$  is Fredholm we can choose  $\epsilon > 0$  such that

$$\alpha(\lambda_\infty - T_\infty) = k(\lambda_\infty - T_\infty) = k(T_\infty) = \alpha(T_\infty) \quad \text{for all } 0 < |\lambda| < \epsilon.$$

This shows that  $\alpha(T_\infty) = \alpha(T)$  and consequently  $N(T) \subset R^\infty(T)$ . Therefore  $T$  is regular.

**Case 2**  $T \in \phi_-(X)$ . Assume that  $T$  is regular. Then  $T' \in \phi_+(X')$  (see Proposition 8) and  $T' \in R(X')$  (see Theorem 13) and thus from Case 1 it follows that  $j(T') = 0$ , so that  $j(T) = 0$ . Consider now that  $T \in \phi_-(X)$  and  $j(T) = 0$ . Then  $T' \in \phi_+(X')$  (see again Proposition 8) and

$j(T') = 0$ . From Case 1,  $T'$  is regular, so that  $N(T') \subset R((T')^n)$  for all  $n \in \mathbb{N}$ . Then we have that

$$R(T)^\perp = N(T' \subset R((T')^n) \subset R((T^n)') \subset (R((T^n)')^\perp)^\top \subset N(T^n)^\perp \quad (\text{by Proposition 5}),$$

which implies that  $N(T^n) \subset (R(T)^\perp)^\top = R(T)$  (as  $R(T)$  is closed).

Consequently,  $T \in R(X)$ , as required.  $\square$

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